FLOW AROUND A SPHERE WITH MATERIAL CROSS FLOW AT LOW REYNOLDS NUMBERS

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UDC 532.516

An analytical solution is carried out for the problem of the flow around a sphere with material cross flow at Reynolds numbers less than 1 and a blowing velocity less than the free stream velocity. The method of asymptotic expansions of Pearson and Proudman is used for the solution. Expressions are obtained for the distribution of the current and velocity component functions as well as for the aerodynamic drag coefficient of the sphere. It is shown that blowing diminishes the sphere drag, where its influence will increase as the Reynolds number grows.

The solution of problems about the flow around, and the heat and mass exchange from, a spherical particle with blowing on the surface at low Reynolds numbers is of definite interest for the analysis of such processes in disperse flows as drying, sublimation, thermal expansion, fuel combustion, heterogeneous reaction with a Stefan stream, etc.

Among the papers in this area [1-8], both [6, 7] should be considered the first systematic investigations in which approximate analytical solutions were obtained for both the hydrodynamic and heat problems, and an experimental confirmation was carried out. However, only the first, Stokes, approximation was found for the velocity field in [6], which does not permit estimation of the influence of blowing on the sphere drag. In the same paper the uniformly valid solution for the heat problem for the whole flow domain was also sought by the essentially classical method of expansion in a small parameter.

Meanwhile, Oseen [9], Proudman and Pearson [10] (for the hydrodynamic problem), and Acrivos and Taylor [11] (for the heat problem but without blowing) showed that the inertial or convective terms become of the same order as the molecular transport terms far from the sphere, and hence, the ordinary method of decomposition in a small parameter yields a known error since, in a second approximation, it does not permit strict satisfaction of the boundary conditions at infinity nor obtaining a single exact solution in the whole domain from $r = 1$ to $r = \infty$.

The purpose herein is a more correct determination of the velocity field and aerodynamic drag coefficient of the sphere for the case under consideration. An attempt is made to solve the problem of flow around a sphere with uniform blowing at $R < 1$ ($R = aU_\infty \nu^{-1}$, a is the coefficient of kinematic viscosity) in at least a second approximation by using the analytical method developed in [10], which is apparently the most rigorous of existing methods. Henceforth, we keep in mind the use of results herein also for the more exact solution of the corresponding heat problem by using a method analogous to that proposed in [11].

The fundamental equation for the stream function in the notation from [10] is

$$\frac{1}{r^2}\frac{\partial(\psi, D^2\psi)}{\partial(r,\mu)} + \frac{2}{r^2}D^2\psi\left(\frac{\mu}{1-\mu^2}\frac{\partial\psi}{\partial r} + \frac{1}{r}\frac{\partial\psi}{\partial\mu}\right) = \frac{1}{R}D^4\psi \tag{1}$$

$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial\mu^2}$$

where D^2 is the Stokes operator and $\mu \equiv \cos\theta$, $\theta = 0$ is the free stream direction.

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 103-109, May-June, 1972. Original article submitted October 7, 1971.

According to [10], its solution is sought in the form of two different approximate expansions for the same stream function, which is a rigorous but unknown solution in the whole flow domain. One of them ψ should satisfy the boundary conditions on the surface exactly and be valid only for the internal or Stokes flow domain near the sphere. It is represented as

$$\psi = \sum_{n=0}^{\infty} f_n(R)\,\psi_n(r, \mu) \tag{2}$$

under the condition that $f_{n+1}/f_n \to 0$, if $R \to 0$, and can be found directly from (1).

The second should satisfy the boundary conditions at infinity and be valid only in the external or Oseen domain at large distances from the sphere. By the change of variable

$$\rho = Rr, \quad \Psi = R^2\psi \tag{3}$$

(1) is hence converted into the following:

$$\frac{1}{\rho^2}\frac{\partial(\Psi, D_\rho^2\Psi)}{\partial(\rho, \mu)} + \frac{2}{\rho^2} D_\rho^2\Psi\left(\frac{\mu}{1-\mu^2}\frac{\partial\Psi}{\partial\rho} + \frac{1}{\rho}\frac{\partial\Psi}{\partial\mu}\right) = D_\rho^4\Psi \tag{4}$$

and the solution is also sought as the series

$$\Psi = \sum_{n=0}^{\infty} F_n(R)\,\Psi_n(\rho, \mu) \tag{5}$$

where $F_{n+1}/F_n \to 0$ as $R \to 0$.

When there is blowing and the constants of the injected material and the free stream are identical, these equations remain valid, but the boundary conditions to which rot V = $(r \sin\theta)^{-1}D^2\psi$ belongs change. If (3) is taken into account, and the radial velocity of blowing through the surface is denoted by V^*, they then are

$$V_r = k = \frac{V^*}{U_\infty}, \quad V_\theta = 0 \quad \text{for} \quad r = 1$$

or

$$\frac{\partial\psi}{\partial\mu} = -k, \quad \frac{\partial\psi}{\partial r} = 0$$

$$V_r = \mu, \quad V_\theta = -(1-\mu^2)^{1/2}, \quad \text{rot}\, V = 0 \quad \text{for} \quad r \to \infty \tag{6}$$

or

$$-\frac{1}{\rho^2}\frac{\partial\Psi}{\partial\mu} = \mu, \quad \frac{1}{\rho}\frac{\partial\Psi}{\partial\rho} = 1-\mu^2, \quad \frac{D_\rho^2\Psi}{\rho} = 0 \tag{7}$$

Since not one but two expansions are sought, the boundary conditions (6) and (7) are insufficient for the solution. However, as has been said, both expansions are only different representations of the same stream function which are homogeneously valid in the whole flow domain. Hence, the missing boundary conditions result from the condition of the identity of the asymptotic continuations of either expansion in some intermediate domain. Similarly for rot V (or curl V).

This matching condition for the stream functions is

$$\underset{r\to\infty}{\psi} = \underset{\rho\to 0}{R^{-2}\Psi} \tag{8}$$

and for curl V

$$\underset{r\to\infty}{\frac{1}{r}D_r^2\psi} = \underset{\rho\to 0}{\frac{R}{\rho}D_\rho^2\Psi} \tag{9}$$

For the case k < 1 under consideration, the velocity should nowhere exceed the order of the free stream velocity. Hence, the principal term in the interior expansion (2) should be independent of R, and we can assume $f_0(R) = 1$. Then, (1) becomes for the principal term

$$D_r^4\psi_0 = 0 \tag{10}$$

and its solution which remains finite for $\mu = \pm 1$, is

$$D_r^2\psi_0 = \sum_{n=0}^{\infty}\left(\frac{A_n}{r^n} + B_n r^{n+1}\right) Q_n(\mu) \tag{11}$$

$$Q_n(\mu) = \int_{-1}^{\mu} P_n(\mu)\, d\mu$$

where $P_n(\mu)$ is the Legendre polynomial of the first kind.

The influence of blowing in the Oseen domain is vanishingly small, and hence, the principal term of the expansion (5) can be found from the unperturbed flow condition at infinity, as in [10]:

$$\Psi_0 = {}^1/_2\rho^2(1-\mu^2) \tag{12}$$

where $F_0(R) = 1$, as follows from the selection of the Oseen coordinates (3). On the other hand, $D_\rho^2\Psi_0 = 0$, in the unperturbed stream, and the matching condition (9) will hold if (11) is represented in the coordinates (3) and it is assumed that $F_1(R) = R$:

$$\sum_{n=0}^{\infty}\left(\frac{A_n R^{n+1}}{\rho^{n+1}} + \frac{B_n \rho^n}{R^n}\right)_{r\to\infty} Q_n(\mu) = O(R^2) \tag{13}$$

Condition (13) is satisfied only if $A_0 = 0$ and $B_n = 0$, i.e., it follows from (11) that

$$D_r^2\psi_0 = \sum_{n=1}^{\infty}\frac{A_n}{r^n} Q_n(\mu) \tag{14}$$

The general solution of (14) is

$$\psi_0 = \sum_{n=0}^{\infty}\left[\frac{C_n}{r^n} + D_n r^{n+1} - \frac{A_n}{2(2n-1)r^{n-2}}\right] Q_n(\mu), \qquad A_0 = 0 \tag{15}$$

and the solution satisfying the boundary conditions on the surface (6) is

$$\psi_0 = \sum_{n=0}^{\infty}\left[\frac{2n-1}{2r^n} + r^{n+1} - \frac{2n+1}{2r^{n-2}}\right] D_n Q_n(\mu) - k(1+\mu) \tag{16}$$

In order to match the solutions (12) and (16) in conformity with the condition (8), only the higher terms remain in (16) as $r \to \infty$, and it is represented, as before, in the coordinates (3)

$$\sum_{n=1}^{\infty} D_n \left(\frac{\rho^{n+1}}{R^{n-1}} Q_n(\mu) - k(1+\mu)R^2 + \ldots\right)_{r\to\infty} = \left(\frac{\rho^2}{2}(1-\mu^2)\right)_{\rho\to 0} \tag{17}$$

It hence, follows directly that $D_n = 0$ for $n \neq 1$, and $D_1 = -1$, since $Q_1(\mu) = -(1-\mu^2)/2$. Therefore, the first approximation for the interior domain is

$$\psi_0 = {}^1/_4\,(2r^2 - 3r + {}^1/r)(1-\mu^2) - k(1+\mu) \tag{18}$$

$$D_r^2\psi_0 = \frac{3}{2r}(1-\mu^2) \tag{19}$$

As in [6], it is a simple superposition of the radial velocity field, when there is only blowing, on the Stokes distribution without blowing.

For all the subsequent approximations, the boundary conditions in both domains are

$$\frac{\partial\psi_n}{\partial\mu}=0,\quad \frac{\partial\psi_n}{\partial r}=0,\quad \frac{1}{\rho^2}\frac{\partial\Psi_n}{\partial\mu}=0,\quad \frac{1}{\rho}\frac{\partial\Psi_n}{\partial\rho}=0,\quad \frac{D_\rho^2\Psi_n}{\rho}=0 \tag{20}$$

and the construction of the approximations is made as follows.

The expansion of (4) in the small parameter R for the second member of the expansion taking account of the form of Ψ_0 yields an equation analogous to the known Oseen equation

$$\frac{1}{\rho}(1-\mu^2)\frac{\partial D_\rho^2\Psi_1}{\partial\mu}+\mu\frac{\partial D_\rho^2\Psi_1}{\partial\rho}=D_\rho^4\Psi_1 \tag{21}$$

Its solution for $D_\rho^2\Psi_1$ satisfying the boundary conditions (20) has been presented in [10] and can be represented as

$$D_\rho^2\Psi_1=e^{-\rho(1-\mu)/2}\sum_{n=1}^{\infty}B_n\sum_{s=0}^{n}\frac{(n+s)!}{s!\,(n-s)!\,\rho^s}Q_n(\mu) \tag{22}$$

As is seen from (13), the higher members of the solution (22) should be matched as $\rho\to0$ with the expression for $D_r^2\psi_0$ as $r\to\infty$ by means of the condition (9)

$$-\frac{3R^2}{\rho^2}Q_1(\mu)\Big|_{r\to\infty}=F_1(R)\,R\sum_{n=1}^{\infty}B_n\frac{2n!}{\rho^{n+1}}Q_n(\mu)\Big|_{\rho\to0} \tag{23}$$

This relationship is satisfied if $B_1=-3/2$, $B_n=0$ for $n\neq1$ and $F_1(R)=R$. Therefore

$$D_\rho^2\Psi_1=\frac{3}{4}\left(1+\frac{2}{\rho}\right)(1-\mu^2)\,e^{-\rho(1-\mu)/2} \tag{24}$$

The general solution of this equation, which satisfies the boundary conditions at infinity for the second approximation, is

$$\Psi_1=C_0\rho(1+\mu)+\sum_{n=0}^{\infty}\frac{D_n}{\rho^n}Q_n(\mu)+\frac{3}{2}(1+\mu)\,e^{-\rho(1-\mu)/2} \tag{25}$$

It is seen from the matching condition that all the members of the Stokes expansion for $r\to\infty$ should enter into the Oseen expansion for $\rho\to0$ and conversely. Hence, the higher members of (25) should be matched for $\rho\to0$ with the second member in the expression for ψ_0 since the first member has already been matched:

$$-\frac{3}{4}\frac{\rho}{R}(1-\mu^2)+\ldots\Big|_{r\to\infty}=\frac{1}{R}\Big[C_0\rho(1+\mu)+D_0(1+\mu)+\sum_{n=1}^{\infty}\frac{D_n}{\rho^n}Q_n(\mu)+\frac{3}{2}(1+\mu)-\frac{3}{4}\rho(1-\mu^2)\Big]+\ldots\Big|_{\rho\to0} \tag{26}$$

Comparison of the terms shows that

$$C_0=0,\ D_0=-3/2,\ D_n=0\quad \text{for } n\neq0$$

Therefore, the expression for the second member of the Oseen expansion is

$$\Psi_1=-3/2\,(1+\mu)\,[1-e^{-\rho(1-\mu)/2}] \tag{27}$$

As can be seen from the construction, the influence of blowing upon the solution in the Oseen domain under the constraints made $k<1$ is only perceptible in the third member of the expansion.

The second member of the interior expansion is constructed analogously. Under the assumption that $f_1(R)=R$, and after substituting the expression for ψ_0 in (1), the equation for ψ_1 acquires the following form:

$$\frac{9}{2r^3}\left(2r-3+\frac{1}{r^2}\right)Q_2(\mu)+\frac{9k}{r^4}Q_1(\mu)=D_r^4\psi_1 \tag{28}$$

and its solution for $D_r^2\psi_1$ is

$$D_r^2\psi_1 = \frac{9k}{4r^2}Q_1(\mu) - \frac{3}{2}\left(1 - \frac{9}{4r} - \frac{1}{2r^3}\right)Q_2(\mu) + \sum_{n=0}^{\infty}\left(\frac{A_n}{r^n} + B_n r^{n+1}\right)Q_n(\mu) \tag{29}$$

By means of the condition (9) for $r \to \infty$, this solution is matched to the expression for $D_\rho^2\Psi_1$ for $\rho \to 0$ and it is here taken into account that the highest member in the latter has already been matched

$$\frac{R^2}{\rho}\left[-\frac{3}{2}Q_2(\mu) + A_0Q_0(\mu) + \sum_{n=1}^{\infty}\frac{A_nR^n}{\rho^n}Q_n(\mu) + \sum_{n=0}^{\infty}B_n\frac{\rho^{n+1}}{R^{n+1}}Q_n(\mu)\right]_{r\to\infty} = \left[\frac{3}{4}\mu + \ldots\right]_{\rho\to 0}(1-\mu^2)\frac{R^2}{\rho} \tag{30}$$

This is satisfied if $A_0 = 0$ and $B_n = 0$, i.e.,

$$D_r^2\psi_1 = \frac{9k}{4r^2}Q_1(\mu) - \frac{3}{2}\left(1 - \frac{9}{4r} - \frac{1}{2r^3}\right)Q_2(\mu) + \sum_{n=1}^{\infty}\frac{A_n}{r^n}Q_n(\mu) \tag{31}$$

The solution of this equation can be represented as

$$\psi_1 = -\frac{9k}{8}Q_1(\mu) + \frac{3}{16}\left(2r^2 - 3r - \frac{1}{r}\right)Q_2(\mu) - \sum_{n=0}^{\infty}\frac{A_n}{2(2n-1)r^{n-2}}Q_n(\mu) + \sum_{n=0}^{\infty}\left[\frac{C_n}{r^n} + D_nr^{n+1}\right]Q_n(\mu) \tag{32}$$

or after satisfying the boundary conditions on the surface for ψ_1

$$\begin{aligned}\psi_1 = \sum_{n=1}^{\infty}\left[\frac{2n-1}{2r^n} + r^{n+1} - \frac{2n+1}{2r^{n-2}}\right]D_nQ_n(\mu) + \\ + \frac{9k}{16}\left(r - 2 + \frac{1}{r}\right)Q_1(\mu) + \frac{3}{16}\left(2r^2 - 3r + 1 - \frac{1}{r} + \frac{1}{r^2}\right)Q_2(\mu)\end{aligned} \tag{33}$$

The matching now is according to condition (8) with the expression for Ψ_1, from which the members matched earlier have been eliminated

$$\left[\sum_{n=1}^{\infty}\frac{D_n\rho^{n+1}}{R^{n-1}}Q_n(\mu) + \frac{3}{8}\rho^2Q_2(\mu) + \ldots\right]_{r\to\infty} = \left[-\frac{3}{8}\rho^2Q_1(\mu) + \frac{3}{8}\rho^2Q_2(\mu) + \ldots\right]_{\rho\to 0} \tag{34}$$

Condition (34) is satisfied if $D_1 = -3/8$, $D_n = 0$ for $n \neq 1$. The assumption that $f_1(R) = R$ is verified simultaneously.

We finally have

$$\begin{aligned}\psi_1 = -\frac{3}{8}\left(r^2 - \frac{3}{2}r + \frac{1}{2r}\right)Q_1(\mu) + \\ + \frac{3}{16}\left(2r^2 - 3r + 1 - \frac{1}{r} + \frac{1}{r^2}\right)Q_2(\mu) + \frac{9k}{16}\left(r - 2 + \frac{1}{r}\right)Q_1(\mu)\end{aligned} \tag{35}$$

$$D_r^2\psi_1 = -\frac{9}{8}\left(\frac{1+k}{r} - \frac{2k}{r^2}\right)Q_1(\mu) - \frac{3}{2}\left(1 - \frac{9}{4r} + \frac{3}{4r^2} - \frac{1}{2r^3}\right)Q_2(\mu) \tag{36}$$

As should have been expected, the first two members in (35) are analogous to those obtained in [10], and the last member takes account of interaction between the blowing and the free stream.

If the velocity profiles are determined by means of the two members found for the expansions (5) and (2), then, an expression for the pressure distribution P can be found by using the Navier–Stokes equation for the r-th component of the momentum.

In the Oseen domain ($\rho = Rr$) this distribution which satisfies the condition

$$P_{\rho\to\infty} = P_0 / \rho_0 U_\infty^2$$

at infinity will be

$$P^* = \frac{P_0}{\rho_0 U_\infty^2} - \frac{3}{2}\frac{R}{\rho^2}\cos\theta \tag{37}$$

where ρ_0 is the density of the medium, and its matched pressure distribution in the interior domain is

$$P_* = \frac{P_0}{\rho_0 U_\infty^2} - \frac{3\cos\theta}{2Rr^2} - \frac{9\cos\theta}{16r^2} - \frac{k^2}{2r^4} + \\ + \left(-\frac{25}{16r^2} + \frac{9}{4r^3} - \frac{1}{2r^5}\right) k\cos\theta + \left(\frac{9}{16r^2} - \frac{7}{8r^3} + \frac{15}{16r^4} - \frac{1}{8r^6}\right)\cos^2\theta + \\ + \left(-\frac{9}{16r^2} + \frac{7}{16r^3} - \frac{3}{16r^4} - \frac{1}{32r^6}\right)\sin^2\theta \tag{38}$$

From (38) the pressure on the sphere surface equals

$$P_{r=1} = \frac{P_0}{\rho_0 U_\infty^2} - \frac{3\cos\theta}{2R} - \frac{9\cos\theta}{16} + \frac{\cos^2\theta}{2} - \frac{11\sin^2\theta}{32} - \frac{k^2}{2} + \frac{3k\cos\theta}{16} \tag{39}$$

The coefficient of aerodynamic drag of the sphere in the customary representation $C_f = 2F/\rho_0 U_\infty^2$ (F is the drag) corresponding to the pressure profiles (39) and the tangential stresses on the surface determined from the expansion (2) is

$$C_f = \frac{24}{R^*}\left(1 + \frac{3}{16}R^* - \frac{7k}{48}R^*\right), \qquad R^* = 2R \tag{40}$$

The first two members are the known Stokes expression with the Oseen correction, and the last member characterizes the influence of the blowing and shows that this influence grows towards a diminution in sphere drag as the Reynolds number increases.

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